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The time response of structures with bounded parameters and interval initial conditions

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ABSTRACT

Uncertainty plays an important role in the performance of structures. In this paper, we focus on the dynamic response of structures with bounded parameters and interval initial conditions, and present a new method to determine the supremum and infimum of the time response. The method is based on the vertex solution theorem for the first-order deviation of the dynamic response from its central value and avoids interval extension problems present in current methods, where the length of the interval increases significantly due to the intermediate calculations. The method is more accurate than existing perturbation methods and provides tighter bounds on the response. The approach neglects the second-order terms in the equation of motion, and care should be exercised when the parameter variations are large. The other advantage of this method is its ability to solve problems with uncertainties in the initial conditions.

1. Introduction

Dynamic response simulations are required in the design and analysis of structures for many engineering problems. However, many types of uncertainties exist in dynamic systems, related to the geometry (e.g. the dimensions of a beam section), the material characteristics (e.g. Young's modulus, shear modulus, Poisson's ratio or mass density), and the exterior environment (e.g. thermal properties or exterior loads). The analysis of uncertainty plays an important role in determining the adequate performance of structures.

Many researchers [1–4] have investigated the dynamic response of structures with probabilistic parameters. However, these probabilistic analysis approaches demand significant knowledge about the uncertain parameters that may not be available easily in practice. Thus, an alternative, non-probabilistic conceptual framework [5,6] based on interval mathematics [7,8] arose, in which only the bounds of uncertain parameters are required, and knowledge of the probabilistic distributions is not necessary. Most of the interval analysis for structures has considered the static response [9,10], the eigenvalue problem [11,12], or the frequency response [13,14]. However, the uncertainty analysis of the time response of a structure has received little attention [6]. Qiu and Wang [15,16] and Zhang et al. [17] presented non-probabilistic interval analysis methods to estimate the range of the dynamic response of structures, based on a Taylor series expansion. The disadvantages of this method are the increase in the response intervals due to interval extension arising from the intermediate calculations, and the requirement to calculate the first derivatives of the response with respect to the uncertain parameters. Furthermore, the available methods are only able to solve problems with uncertain structural parameters, whereas often there will also be uncertainties in the initial conditions.

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There are various ways of defining the uncertainty [5]. This paper assumes that the individual parameters are within an interval, and so the set of parameters is defined as a hyperrectangle. The convex set of parameters is then propagated to give a set of responses that is not necessarily convex. Often the hyperrectangle of smallest volume that contains the set of responses is chosen to represent the response uncertainty. The uncertain parameters may also be defined by using ellipsoids, and a response ellipsoid of the smallest volume may be defined that contains the responses. For small levels of uncertainty a first-order perturbation analysis may be sufficient, and in this case the response ellipsoid may be calculated directly [18].

In this paper, a new method is presented to determine the supremum and infimum of the dynamic response with bounded parameters and interval initial conditions. The method is based on the vertex solution theorem for the first-order deviation of the dynamic response from its central value and avoids interval extension problems present in current methods, where the length of the interval increases significantly due to the intermediate calculations. The method is more accurate than existing perturbation methods and provides tighter bounds on the response. The approach neglects the second-order terms in the equation of motion, and care should be exercised when the parameter variations are large. In Section 2, we formulate the problem and in Section 3 present the first-order perturbation of the structural dynamic response problem. The method to calculate the exact bounds of the deviation in the dynamic response by the vertex solution theorem is presented, and the supremum and infimum of the dynamic responses is obtained in Section 4. Examples of interval dynamic problems are used to illustrate the application of the proposed method in Section 5, and the results are compared to the perturbation method. The examples consist of a two degree of freedom discrete system, a two-dimensional truss structure with six nodes and eight elements, and a rotor-disc system with interval initial conditions.

2. Problem formulation

Consider the equation of motion of a linear dynamic system [19] with n degrees of freedom and viscous damping given by

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively, and $\mathbf{f}(t)$ is the external load vector. The structural matrices and external load depend on the uncertain parameter vector $\mathbf{a}=(a_j)$, $j=1,2,\ldots,m$, where the bracket notation means that a_j is the jth element of the vector $\mathbf{a}$. Thus,

$$\mathbf{M}=\mathbf{M}(\mathbf{a}),\ \mathbf{C}=\mathbf{C}(\mathbf{a}),\ \mathbf{K}=\mathbf{K}(\mathbf{a}),\ \mathbf{f}(\mathbf{t})=\mathbf{f}(\mathbf{a},t). \tag{2}$$

Suppose that the structural parameters are uncertain, but are constrained to lie within an interval, given by

$$\mathbf{a}\in\mathbf{a}^I=(a_j^I),\quad a_j\in a_j^I=[\underline{a}_j,\overline{a}_j],\quad j=1,2,\ldots,m, \tag{3}$$

where $\underline{a}_j$ and $\bar{a}_j$ denote the ends of the interval for the jth parameter, a_j. Thus the mass, damping and stiffness matrices are interval matrices, and the external load is an interval vector.

The assumption made in this paper is that the uncertain parameters are fixed during the simulation of the dynamic response. In practice even if the parameters change the time constants will be much larger than those corresponding to the modes of interest. For example the thermal inertia of a highway bridge will cause the structural matrices to change very slowly, or the change in mass due to fuel consumption in an aircraft is likely to be slow. Thus, the structural parameters are assumed to be time independent, although they are unknown and subject to the interval constraint conditions, Eq. (3).

The initial conditions of Eq. (1) are assembled into the vector $\mathbf{y}(0)=\mathbf{y}_0=\begin{Bmatrix}\mathbf{x}(0)\\ \dot{\mathbf{x}}(0)\end{Bmatrix}$. Suppose that some components of this initial condition vector, $\mathbf{y}_0$, are also uncertain but are constrained to lie within an interval. An uncertain initial condition parameter vector is defined as $\mathbf{b}=(b_k)$, whose elements are the uncertain components of the initial condition vector, $\mathbf{y}_0$, given by

$$\mathbf{b}\in\mathbf{b}^I=(b_k^I),\quad b_k\in b_k^I=[\underline{b}_k,\overline{b}_k],\quad k=1,2,\ldots,l,\quad 0\le l\le 2n \tag{4}$$

where l denotes the number of uncertain components. The initial condition vector is then a linear function of the uncertain parameter vector $\mathbf{b}$, as

$$\mathbf{y}(0)=\mathbf{y}_0(\mathbf{b})=\sum_{k=1}^{l}b_k\mathbf{y}_{0k} \tag{5}$$

for some fixed vectors $\mathbf{y}_{0k}$. The solution to the equations of motion with bounded parameter uncertainties is a set, and this set may be expressed as

$$\Gamma=\left\{\mathbf{x}(t):\mathbf{M}(\mathbf{a})\ddot{\mathbf{x}}(t)+\mathbf{C}(\mathbf{a})\dot{\mathbf{x}}(t)+\mathbf{K}(\mathbf{a})\mathbf{x}(t)=\mathbf{f}(\mathbf{a},t),\quad \mathbf{y}(0)=\begin{Bmatrix}\mathbf{x}(0)\\ \dot{\mathbf{x}}(0)\end{Bmatrix}=\mathbf{y}_0(\mathbf{b}),\quad \mathbf{a}\in\mathbf{a}^I,\quad \mathbf{b}\in\mathbf{b}^I\right\}. \tag{6}$$

The computation of this dynamic response set, in general, is extremely difficult. The solution set Γ has a very complicated region and may not be convex. Note also that the boundary of Γ will not be formed by a single actual response. Taking this into account, one has to determine a closed convex interval solution set $\mathbf{x}^I(t)=[\underline{\mathbf{x}}(t), \bar{\mathbf{x}}(t)]$, that has the smallest volume, but which encloses all possible values of the dynamic response $\mathbf{x}(t)\in\Gamma$. This response interval is time dependent, and $\bar{\mathbf{x}}(t)=(\bar{x}_i(t))$ and $\underline{\mathbf{x}}(t)=(\underline{x}_i(t))$ are the supremum and infimum vectors of the dynamic response vector, $\mathbf{x}(t)=(x_i(t))$, respectively. The supremum and infimum vectors defining the interval dynamic response can be expressed as

$$\bar{x}_i(t) = \max_{\mathbf{x}\in\Gamma}\{|x_i(t)|\} \tag{7}$$

and

$$\underline{x}_i(t) = \min_{\mathbf{x}\in\Gamma}\{|x_i(t)|\}. \tag{8}$$

3. First-order perturbation of the structural dynamic response

Obtaining the supremum and infimum vectors in Eqs. (7) and (8) is very difficult. Hence, we will consider the perturbed problem, obtained from Eq. (1) as

$$(\mathbf{M}_c + \delta\mathbf{M})\ddot{\mathbf{x}}(t) + (\mathbf{C}_c + \delta\mathbf{C})\dot{\mathbf{x}}(t) + (\mathbf{K}_c + \delta\mathbf{K})\mathbf{x}(t) = \mathbf{f}_c(t) + \delta\mathbf{f}(t), \tag{9}$$

where $\mathbf{M}_c=\mathbf{M}(\mathbf{a}_c)$, $\mathbf{C}_c=\mathbf{C}(\mathbf{a}_c)$, $\mathbf{K}_c=\mathbf{K}(\mathbf{a}_c)$, $\mathbf{f}_c(t)=\mathbf{f}(\mathbf{a}_c,t)$, and $\mathbf{a}_c=(\underline{\mathbf{a}}+\bar{\mathbf{a}})/2$. Here $\mathbf{M}_c$, $\mathbf{C}_c$, $\mathbf{K}_c$, $\mathbf{f}_c(t)$ represent the central dynamic system and $\delta\mathbf{M}$,$\delta\mathbf{C}$,$\delta\mathbf{K}$,$\delta\mathbf{f}(t)$ are the small changes from this central system. The objective is to compute the perturbed structural dynamic response. Consider the perturbation to the dynamic response, given by

$$\mathbf{x}(t) = \mathbf{x}_c + \delta\mathbf{x}, \quad \dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_c + \delta\dot{\mathbf{x}}, \quad \ddot{\mathbf{x}}(t) = \ddot{\mathbf{x}}_c + \delta\ddot{\mathbf{x}}. \tag{10}$$

Substituting Eq. (10) into Eq. (9) and equating zero-order terms, we have

$$\mathbf{M}_c\ddot{\mathbf{x}}_c(t) + \mathbf{C}_c\dot{\mathbf{x}}_c(t) + \mathbf{K}_c\mathbf{x}_c(t) = \mathbf{f}_c(t) \tag{11}$$

with the initial conditions

$$\mathbf{y}_c(0) = \begin{Bmatrix} \mathbf{x}_c(0) \\ \dot{\mathbf{x}}_c(0) \end{Bmatrix} = \mathbf{y}_0(\mathbf{b}_c),\ \text{where } \mathrm{b}_c = (\frac{\underline{\mathbf{b}} + \overline{\mathbf{b}})}{2}. \tag{12}$$

Equating first-order terms,

$$\mathbf{M}_c\delta\ddot{\mathbf{x}} + \mathbf{C}_c\delta\dot{\mathbf{x}} + \mathbf{K}_c\delta\mathbf{x} = \delta\mathbf{f}(t) - \delta\mathbf{M}\ddot{\mathbf{x}}_c - \delta\mathbf{C}\dot{\mathbf{x}}_c - \delta\mathbf{K}\mathbf{x}_c \tag{13}$$

with the initial conditions

$$\delta\mathbf{y}(0) = \begin{Bmatrix} \delta\mathbf{x}_c(0) \\ \delta\dot{\mathbf{x}}_c(0) \end{Bmatrix} = \sum_{k=1}^{l} \delta b_k \mathbf{y}_{0k}, \tag{14}$$

where $\delta b_k=b_k-b_{ck}$, and $\mathbf{b}_c=(b_{ck})$. Thus $\delta b_k\in[-\Delta b_k,\Delta b_k]$, where $\Delta b_k = (\overline{b}_k - \underline{b}_k)/2$.

Often $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{f}(t)$ will be linear functions of the structural parameter vector $\mathbf{a}=(a_j)$, so that

$$\delta\mathbf{M} = \sum_{j=1}^{m}\delta a_j\mathbf{M}_j, \quad \delta\mathbf{C} = \sum_{j=1}^{m}\delta a_j\mathbf{C}_j, \quad \delta\mathbf{K} = \sum_{j=1}^{m}\delta a_j\mathbf{K}_j, \quad \delta\mathbf{f}(\mathbf{a},t) = \sum_{j=1}^{m}\delta a_j\mathbf{f}_j(t), \tag{15}$$

where $\delta a_j=a_j-a_{cj}$, and $\mathbf{a}_c=(a_{cj})$. Thus, $\delta a_j\in[-\Delta a_j,\Delta a_j]$, where $\Delta a_j=(\bar{a}_j-\underline{a}_j)/2$. In other cases Eq. (15) is a good first-order approximation to the structural matrices and the force, where the matrices and vectors are given by

$$\mathbf{M}_j = \frac{\partial\mathbf{M}}{\partial a_j}\bigg|_{a_j=a_{cj}}, \quad \mathbf{C}_j = \frac{\partial\mathbf{C}}{\partial a_j}\bigg|_{a_j=a_{cj}}, \quad \mathbf{K}_j = \frac{\partial\mathbf{K}}{\partial a_j}\bigg|_{a_j=a_{cj}}, \quad \mathbf{f}_j = \frac{\partial\mathbf{f}}{\partial a_j}\bigg|_{a_j=a_{cj}} \tag{16}$$

3.1. The perturbation method of Qiu and Wang

Qiu and Wang [15] presented a perturbation method to estimate the boundary of the perturbed response given by $\Delta\mathbf{x}(t)$, satisfying $|\delta\mathbf{x}(t)|\le\Delta\mathbf{x}(t)$ for all $\delta\mathbf{x}(t)$. The method assumes a perturbed response of the form

$$\delta\mathbf{x}(t) = \sum_{j=1}^{m}\delta a_j\mathbf{X}_j(t), \tag{17}$$

where the $\mathbf{X}_j$ terms are obtained by substituting Eqs. (15) and (17) into Eq. (13) and equating coefficients of δa_j. Thus, the $\mathbf{X}_j(t)$ terms are the solutions of the differential equations

$$\mathbf{M}_c\ddot{\mathbf{X}}_j + \mathbf{C}_c\dot{\mathbf{X}}_j + \mathbf{K}_c\mathbf{X}_j = \mathbf{f}_j(t) - (\mathbf{M}_j\ddot{\mathbf{x}}_c(t) + \mathbf{C}_j\dot{\mathbf{x}}_c(t) + \mathbf{K}_j\mathbf{x}_c(t)), \tag{18}$$

where the right side contains known functions. Finally, the bounds on the response are estimated as

$$\Delta \mathbf{x}(t) = \sum_{j=1}^{m} |\mathbf{X}_j(t)| \Delta a_j \tag{19}$$

and the infimum and supremum of the dynamic response are then $\underline{\mathbf{x}}(t)=\mathbf{x}_0(t)-\Delta\mathbf{x}(t)$ and $\bar{\mathbf{x}}(t)=\mathbf{x}_0(t)+\Delta\mathbf{x}(t)$. It should be noted that taking the absolute values in Eq. (19) will lead to an over-estimate of the width of the response interval. This method is only able to solve problems with uncertain structural parameters but with certain initial conditions.

4. Obtaining the bounds from the perturbed equations

The estimation of the response bounds using the method of Qiu and Wang [15,16] has two approximations, namely the neglecting of second-order terms in Eq. (9) and the interval extension in Eq. (19). The main purpose of this paper is to remove the approximation in the interval extension by calculating exact bounds for the deviation in the dynamic response from the perturbation equations. Note that these bounds define the envelope of possible responses, rather than the bounds on an individual response.

Eq. (13) may be written in state space form as

$$\delta \dot{\mathbf{y}}(t) = \mathbf{A}\delta \mathbf{y}(t) + \mathbf{u}(t) \tag{20}$$

where $\delta\mathbf{y}(t)$ is the state vector of dimension $2n$, given by

$$\delta \mathbf{y}(t) = \begin{Bmatrix} \delta \mathbf{x}(t) \\ \delta \dot{\mathbf{x}}(t) \end{Bmatrix} \tag{21}$$

with the initial condition

$$\delta \mathbf{y}(0) = \begin{Bmatrix} \delta \mathbf{x}(0) \\ \delta \dot{\mathbf{x}}(0) \end{Bmatrix} = \sum_{k=1}^{l} \delta b_k \mathbf{y}_{0k} \text{ for } k = 1, 2, \ldots, l. \tag{22}$$

The state matrix and input vector are

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}_c^{-1}\mathbf{K}_c & -\mathbf{M}_c^{-1}\mathbf{C}_c \end{bmatrix}, \quad \mathbf{u}(t) = \begin{Bmatrix} 0 \\ \mathbf{M}_c^{-1}(\delta\mathbf{f}(t) - \delta\mathbf{M}\ddot{\mathbf{x}}_c(t) - \delta\mathbf{C}\dot{\mathbf{x}}_c(t) - \delta\mathbf{K}\mathbf{x}_c(t)) \end{Bmatrix} \tag{23}$$

where $\mathbf{I}$ is the identity matrix.

Assuming that the mass, damping and stiffness matrices and the external force are given in terms of the parameter vector in the form shown in Eq. (15), then the state input vector may be written in the form

$$\mathbf{u}(t) = \sum_{j=1}^{m} \delta a_j \mathbf{u}_j(t), \tag{24}$$

where

$$\mathbf{u}_j(t) = \begin{Bmatrix} 0 \\ \mathbf{M}_c^{-1}(\mathbf{f}_j(t) - \mathbf{M}_j\ddot{\mathbf{x}}_c(t) - \mathbf{C}_j\dot{\mathbf{x}}_c(t) - \mathbf{K}_j\mathbf{x}_c(t)) \end{Bmatrix}. \tag{25}$$

4.1. *Obtaining the exact bounds of the deviation in the dynamic response*

Solving the state space form given by Eq. (20), and using the superposition principle for linear ordinary differential equations, the solution is

$$\delta \mathbf{y}(\delta\mathbf{a}, \delta\mathbf{b}, t) = \sum_{k=1}^{l} \delta b_k \mathbf{y}_{bk}(t) + \sum_{j=1}^{m} \delta a_j \mathbf{y}_{aj}(t), \tag{26}$$

where

$$\mathbf{y}_{bk}(t) = e^{\mathbf{A}t}\mathbf{y}_{0k} \tag{27}$$

and

$$\mathbf{y}_{aj}(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{u}_j(\tau)\,\mathrm{d}\tau. \tag{28}$$

Since the vectors $\mathbf{y}_{bk}(t)$ and $\mathbf{y}_{aj}(t)$ do not depend on the perturbation in the parameters, $\delta\mathbf{y}(\delta\mathbf{a},\delta\mathbf{b},t)$ is a linear function with respect to the uncertain parameters, δa_j, $j=1,2,\ldots,m$ and δb_k, $k=1,2,\ldots,l$.

Geometrically, the linear constraints given by $\delta a_j \in [-\Delta a_j, \Delta a_j]$ and $\delta b_k \in [-\Delta b_k, \Delta b_k]$ define a convex polyhedron, which is called the feasible region. Since the response deviation, $\delta\mathbf{y}(\delta\mathbf{a},\delta\mathbf{b},t)$, is a linear function of the parameters, the time response also forms a convex region, and the extreme values are thus attained at a vertex of the polyhedron.

Let the set of vertices of the convex region in parameter space be given by

$$\Xi_\delta = \left\{ (\delta\mathbf{a}, \delta\mathbf{b}) : \begin{array}{lll} \delta\mathbf{a} = (\delta a_j), & \text{where } \delta a_j = -\Delta a_j \text{ or } \delta a_j = \Delta a_j, & \text{for } j = 1, \ldots, m \\ \delta\mathbf{b} = (\delta b_k), & \text{where } \delta b_k = -\Delta b_k \text{ or } \delta b_k = \Delta b_k, & \text{for } k = 1, \ldots, l \end{array} \right\} \tag{29}$$

which is equivalent to

$$\Xi = \left\{ \mathbf{a}, \mathbf{b} : \begin{array}{lll} \mathbf{a} = a_j, & \text{where } a_j = -\underline{a}_j \text{ or } a_j = \overline{a}_j, & \text{for } j = 1, \ldots, m \\ \mathbf{b} = b_k, & \text{where } b_k = -\underline{b}_k \text{ or } b_k = \overline{b}_k, & \text{for } k = 1, \ldots, l \end{array} \right\} \tag{30}$$

This set of vertices has 2^{m+l} elements.

Thus the exact the supremum and the infimum of $\delta\mathbf{y}(\delta\mathbf{a},\delta\mathbf{b},t)$ in Eq. (26) may be determined as

$$\overline{\delta\mathbf{y}}(t) = \max_{(\delta\mathbf{a},\delta\mathbf{b})\in\Xi_\delta} \left\{ \sum_{k=1}^{l} \delta b_k \mathbf{y}_{bk}(t) + \sum_{j=1}^{m} \delta a_j \mathbf{y}_{aj}(t) \right\} \tag{31}$$

and

$$\underline{\delta\mathbf{y}}(t) = \min_{(\delta\mathbf{a},\delta\mathbf{b})\in\Xi_\delta} \left\{ \sum_{k=1}^{l} \delta b_k \mathbf{y}_{bk}(t) + \sum_{j=1}^{m} \delta a_j \mathbf{y}_{aj}(t) \right\}. \tag{32}$$

Thus, the exact the supremum and the infimum of the first-order deviation of the dynamic response, $\delta\mathbf{x}$, may be determined by the following expressions, which are equivalent to Eqs. (31) and (32):

$$\overline{\delta\mathbf{x}}(t) = \max_{(\mathbf{a},\mathbf{b})\in\Xi} \left\{ \delta\mathbf{x}(t) : \begin{array}{c} \mathbf{M}_c\delta\ddot{\mathbf{x}} + \mathbf{C}_c\delta\dot{\mathbf{x}} + \mathbf{K}_c\delta\mathbf{x} = \mathbf{f}(\mathbf{a}, t) - \mathbf{M}(\mathbf{a})\ddot{\mathbf{x}}_c - \mathbf{C}(\mathbf{a})\dot{\mathbf{x}}_c(t) - \mathbf{K}(\mathbf{a})\mathbf{x}_c(t) \\ \delta\mathbf{y}(0) = \left\{ \begin{array}{c} \delta\mathbf{x}(0) \\ \delta\dot{\mathbf{x}}(0) \end{array} \right\} = \mathbf{y}_0(b) - \mathbf{y}_0(b_c) \end{array} \right\} \tag{33}$$

and

$$\underline{\delta\mathbf{x}}(t) = \min_{(\mathbf{a},\mathbf{b})\in\Xi} \left\{ \delta\mathbf{x}(t) : \begin{array}{c} \mathbf{M}_c\delta\ddot{\mathbf{x}} + \mathbf{C}_c\delta\dot{\mathbf{x}} + \mathbf{K}_c\delta\mathbf{x} = \mathbf{f}(\mathbf{a}, t) - \mathbf{M}(\mathbf{a})\ddot{\mathbf{x}}_c - \mathbf{C}(\mathbf{a})\dot{\mathbf{x}}_c(t) - \mathbf{K}(\mathbf{a})\mathbf{x}_c(t) \\ \delta\mathbf{y}(0) = \left\{ \begin{array}{c} \delta\mathbf{x}(0) \\ \delta\dot{\mathbf{x}}(0) \end{array} \right\} = \mathbf{y}_0(\mathbf{b}) - \mathbf{y}_0(\mathbf{b}_c) \end{array} \right\} \tag{34}$$

Thus, the forcing terms in the first-order perturbation dynamic equations shown in Eqs. (33) and (34) are obtained from the displacement, velocity and acceleration time-dependent functions at the central values of the uncertain parameters.

We can now give the algorithm flow based on the vertex solution theorem for solving the dynamic response of structures with bounded uncertainties as follows:

- Calculate the central dynamic response, $\mathbf{x}_c(t)$, by solving Eq. (11). The corresponding velocity and acceleration are obtained from this displacement response.
- Define the set of parameter vertices given by Eq. (30)
- For each parameter vertex, $(\mathbf{a}^\nu,\mathbf{b}^\nu)\in\Xi$, obtain $\delta\mathbf{x}^\nu(\mathbf{a}^\nu,\mathbf{b}^\nu,t)$ by solving the differential equation (Eq. (13)),

$$\mathbf{M}_c\delta\ddot{\mathbf{x}}^\nu + \mathbf{C}_c\delta\dot{\mathbf{x}}^\nu + \mathbf{K}_c\delta\mathbf{x}^\nu = \mathbf{f}(\delta\mathbf{a}^\nu, t) - \mathbf{M}(\delta\mathbf{a}^\nu)\ddot{\mathbf{x}}_c - \mathbf{C}(\delta\mathbf{a}^\nu)\dot{\mathbf{x}}_c(t) - \mathbf{K}(\delta\mathbf{a}^\nu)\mathbf{x}_c(t)$$

  with the initial condition $\delta\mathbf{y}^\nu(0) = \left\{ \begin{array}{c} \delta\mathbf{x}^\nu(0) \\ \delta\dot{\mathbf{x}}^\nu(0) \end{array} \right\} = \mathbf{y}_0(\mathbf{b}^\nu) - \mathbf{y}_0(\mathbf{b}_c)$.
- Then, at each time of interest,

$$\underline{\delta\mathbf{x}}(t) = \min_{(\mathbf{a}^\nu,\mathbf{b}^\nu)\in\Xi} \{\delta\mathbf{x}^\nu(\mathbf{a}^\nu, \mathbf{b}^\nu, t)\}, \quad \overline{\delta\mathbf{x}}(t) = \max_{(\mathbf{a}^\nu,\mathbf{b}^\nu)\in\Xi} \{\delta\mathbf{x}^\nu(\mathbf{a}^\nu, \mathbf{b}^\nu, t)\}.$$

- Estimate the supremum and infimum of the dynamic responses as

$$\underline{\mathbf{x}}(t) = \mathbf{x}_c(t) + \underline{\delta\mathbf{x}}(t), \quad \overline{\mathbf{x}}(t) = \mathbf{x}_c(t) + \overline{\delta\mathbf{x}}(t).$$

4.2. The direct vertex method

The vertex method proposed in this paper requires the numerical integration of Eqs. (33) and (34). The forcing for the differential equation for the response deviation for a given parameter vertex is a function of the response at the central parameter values, $\mathbf{x}_c(t)$. However, this central response is usually obtained by numerical integration, and hence the solution to Eqs. (33) and (34) requires either a fixed time step or interpolation of the central response. One alternative is to reintroduce the second-order terms neglected earlier. Although it may seem odd reintroducing terms previously neglected, one key reason for introducing the response deviation is that this allowed the proof of the vertex theorem. Although the vertex theorem is not proved for the original differential equation, if the second-order terms are small then we can obtain the response bounds by the *direct vertex method* as

$$\overline{\mathbf{x}}(t) = \max_{(\mathbf{a},\mathbf{b})\in\Xi} \left\{ \mathbf{x}(t) : \mathbf{M}(\mathbf{a})\ddot{\mathbf{x}} + \mathbf{C}(\mathbf{a})\dot{\mathbf{x}} + \mathbf{K}(\mathbf{a})\mathbf{x} = \mathbf{f}(\mathbf{a},t), \mathbf{y}(0) = \begin{Bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{Bmatrix} = \mathbf{y}_0(\mathbf{b}) \right\} \tag{35}$$

and

$$\underline{\mathbf{x}}(t) = \min_{(\mathbf{a},\mathbf{b})\in\Xi} \{ \mathbf{x}(t) : \mathbf{M}(\mathbf{a})\ddot{\mathbf{x}} + \mathbf{C}(\mathbf{a})\dot{\mathbf{x}} + \mathbf{K}(\mathbf{a})\mathbf{x} = \mathbf{f}(\mathbf{a},t), \mathbf{y}(0) = \begin{Bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{Bmatrix} = \mathbf{y}_0(\mathbf{b}) \}. \tag{36}$$

Comparing the bounds given by Eqs. (33) and (34) to those given by Eqs. (35) and (36) gives some indication of the size of the second-order terms.

5. Numerical examples

5.1. Discrete mass–spring–damper system

To illustrate the effectiveness of the vertex solution theorem for the interval dynamic problem, we consider the two degree of freedom mass–spring–damper system shown in Fig. 1. The centre values of the stiffness, mass and damping parameters are:

$$k_1^c = 1.0\times10^3\,\text{N/m}, k_2^c = 1.0\times10^3\,\text{N/m}, k_3^c = 4.0\times10^3\,\text{N/m}, m_1^c = 1.0\,\text{kg}, m_2^c = 1.2\,\text{kg}, c_1^c = 4\,\text{kg/s},$$
$$c_2^c = 5\,\text{kg/s}, c_3^c = 5\,\text{kg/s}.$$

The centre function of the force vector is

$$\mathbf{f}^c(t) = \begin{Bmatrix} F_1^c(t) \\ F_2^c(t) \end{Bmatrix} = \begin{Bmatrix} 60 \\ 150 \end{Bmatrix} te^{1-t/\kappa}\,\text{N, where } \kappa = 0.02.$$

The initial conditions are $\mathbf{x}_0$=$\mathbf{x}(0)$=0 and $\dot{\mathbf{x}}_0$=$\dot{\mathbf{x}}(0)$=0.

The deviations in the parameters are

$$\Delta k_2 = 0.02k_2^c,\ \Delta c_3 = 0.015c_3^c,\ \Delta m_1 = 0.04m_1^c,\ \Delta m_2 = 0.03m_2^c,\ \Delta f(t) = 0.05f^c(t)$$

The eigenvalues of this central system are $-2.997\pm40.82i$ and $-5.670\pm66.68i$. Fig. 2 shows the central value, supremum and infimum of the external forces. The central value, infimum and supremum of the dynamic response, computed by the vertex solution theorem, are given in Fig. 3. For comparison the deviations of the dynamic response by vertex solution theorem and by the perturbation method [14] are shown graphically in Figs. 4 and 5, and numerically in Tables 1 and 2. From these results, we can see that the vertex solution theorem proposed in this paper yields narrower bounds than those produced by the perturbation method. The reason is that the vertex solution theorem solves the exact boundary of the first-order deviation of the dynamic response and avoids interval extension. The only approximation in this method is that the second-order terms are neglected.

Figs. 6 and 7 show the comparison of the bounds in the response obtained by the vertex method, based on the response deviation, and the direct vertex method, based on the original differential equations. The maximum error is under 5% and shows that the second-order terms are small in this case.

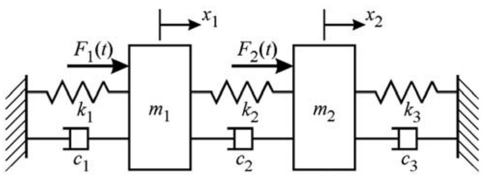

Fig. 1. The two degree of freedom spring–mass–damper system.

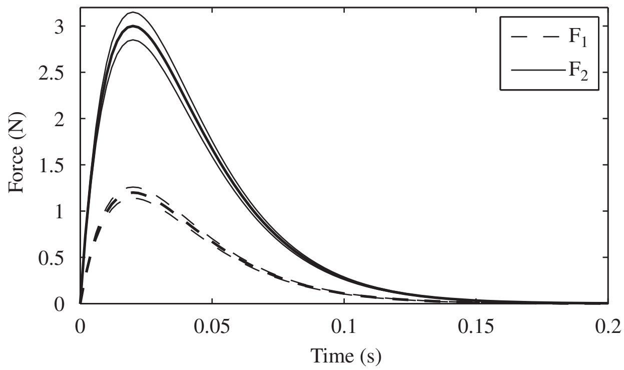

Fig. 2. The external forces for the discrete example.

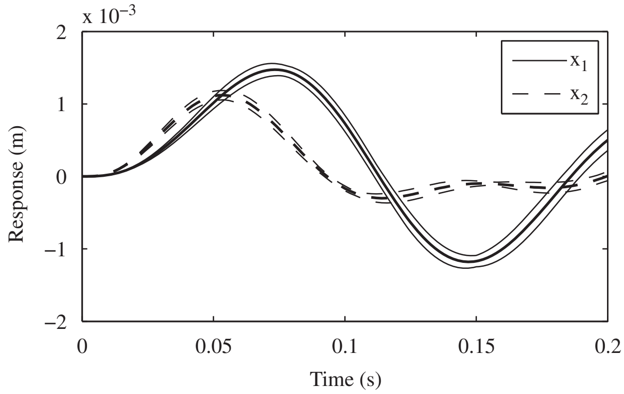

Fig. 3. The central value, infimum and supremum of the dynamic response by the vertex solution theorem for the discrete example.

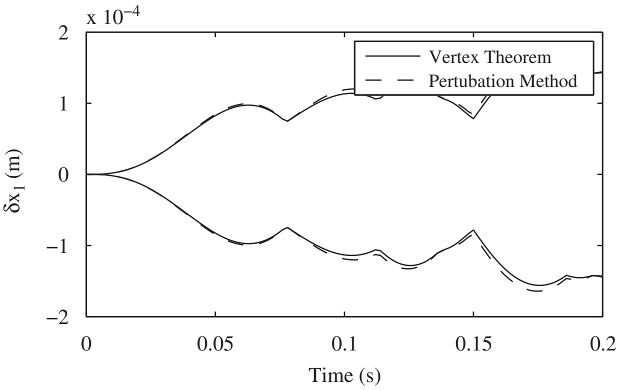

Fig. 4. Comparison of bounds of δx_1 by the vertex solution theorem and the perturbation method for the discrete example.

5.2. Eight-bar truss

Consider the plane truss shown in Fig. 8, subjected to harmonic sinusoidal excitations $P_1(t)$=500$p$sin(200πt) and $P_2(t)$=800$p$sin(100πt), with initial conditions x_0=x(0)=0 and $\dot{x}_0$=$\dot{x}$(0)=0. The truss is modeled with six nodes and eight elements. The cross-sectional area of element numbers 1, 2, 3 and 4 are equal and given by A_1=A_2=A_3=A_4=$1.0\times10^{-4}\,\mathrm{m}^2$, and for element numbers 5, 6, 7 and 8 are A_5=A_6=A_7=A_8=$1.2\times10^{-4}\,\mathrm{m}^2$. The material has a Poisson's ratio of υ=0.3. Rayleigh damping was assumed so that the damping matrix is given by $\mathbf{C}$=$\alpha\mathbf{M}+\beta\mathbf{K}$, where α=20 s^{-1} and β=0.000016 s. Young's modulus, mass density and harmonic sinusoidal excitation amplitude of the plane truss are uncertain but bounded

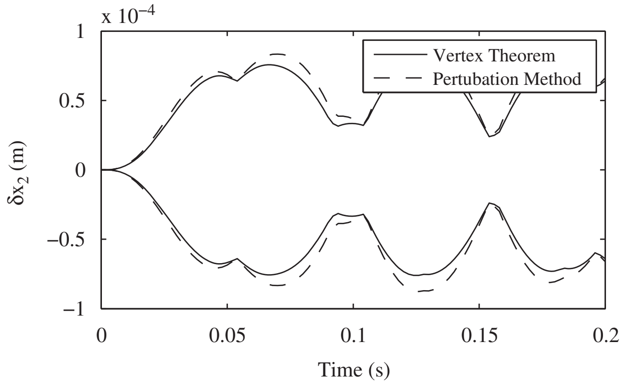

Fig. 5. Comparison of bounds of δx_2 by the vertex solution theorem and the perturbation method for the discrete example.

Table 1
The interval of x_1 by the vertex solution theorem and the perturbation method for the discrete example.

t(s)	x_1^C (m)	$\underline{x}_1^V$ (m)	$\overline{x}_1^V$ (m)	$\underline{x}_1^P$ (m)	$\overline{x}_1^P$ (m)	Δx_1^V (m)	Δx_1^P (m)
0.00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.02	1.34E−04	1.22E−04	1.46E−04	1.22E−04	1.46E−04	1.20E−05	1.21E−05
0.04	6.74E−04	6.17E−04	7.32E−04	6.15E−04	7.33E−04	5.78E−05	5.90E−05
0.06	1.30E−03	1.21E−03	1.40E−03	1.21E−03	1.40E−03	9.65E−05	9.87E−05
0.08	1.43E−03	1.35E−03	1.50E−03	1.35E−03	1.51E−03	7.94E−05	7.99E−05
0.10	7.30E−04	6.17E−04	8.44E−04	6.12E−04	8.49E−04	1.13E−04	1.19E−04
0.12	−3.86E−04	−5.09E−04	−2.63E−04	−5.15E−04	−2.57E−04	1.23E−04	1.29E−04
0.14	−1.12E−03	−1.22E−03	−1.02E−03	−1.23E−03	−1.02E−03	1.04E−04	1.06E−04
0.16	−1.00E−03	−1.13E−03	−8.76E−04	−1.13E−03	−8.67E−04	1.25E−04	1.34E−04
0.18	−2.56E−04	−4.09E−04	−1.02E−04	−4.16E−04	−9.55E−05	1.53E−04	1.60E−04
0.20	5.02E−04	3.59E−04	6.45E−04	3.57E−04	6.46E−04	1.43E−04	1.44E−04

Table 2
The interval of x_2 by the vertex solution theorem and perturbation method for the discrete example.

t (s)	x_2^C (m)	$\underline{x}_2^V$ (m)	$\bar{x}_2^V$ (m)	$\underline{x}_2^P$ (m)	$\overline{x}_2^P$ (m)	Δx_2^V (m)	Δx_2^P (m)
0.00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.02	2.48E−04	2.29E−04	2.67E−04	2.27E−04	2.69E−04	1.91E−05	2.13E−05
0.04	9.21E−04	8.58E−04	9.84E−04	8.53E−04	9.88E−04	6.29E−05	6.73E−05
0.06	1.07E−03	9.99E−04	1.14E−03	9.96E−04	1.15E−03	7.22E−05	7.55E−05
0.08	4.65E−04	4.01E−04	5.29E−04	3.90E−04	5.40E−04	6.40E−05	7.50E−05
0.10	−1.66E−04	−1.99E−04	−1.32E−04	−2.03E−04	−1.29E−04	3.34E−05	3.68E−05
0.12	−2.86E−04	−3.59E−04	−2.12E−04	−3.69E−04	−2.02E−04	7.34E−05	8.34E−05
0.14	−1.35E−04	−1.99E−04	−6.99E−05	−2.08E−04	−6.10E−05	6.46E−05	7.35E−05
0.16	−1.12E−04	−1.46E−04	−7.69E−05	−1.50E−04	−7.29E−05	3.47E−05	3.87E−05
0.18	−1.55E−04	−2.28E−04	−8.18E−05	−2.35E−04	−7.40E−05	7.29E−05	8.07E−05
0.20	6.18E−06	−5.78E−05	7.01E−05	−6.00E−05	7.23E−05	6.40E−05	6.61E−05

parameters, and their interval numbers are: E^I=[190,210] GN/m^2, ρ^I=[7790,7810] kg/m^3 and p^I=[0.98, 1.02] N. The eigenvalues of the truss structure corresponding to the central parameters are shown in Table 3.

The dynamic response at the fourth node of the plane truss in the horizontal direction, computed by the vertex solution theorem, is shown in Fig. 9. The dynamic response of the sixth node in the horizontal and vertical directions, are given in Figs. 10 and 11, respectively.

5.3. A simple rotating machine

Consider the model of a rotor–disc system, 1.5 m long with bearings at 0.0 and 1.5 m, shown in Fig. 12. The bearings are short in that they present insignificant angular stiffness to the shaft but they present finite translational stiffness. The shaft is 25 mm in diameter and the disk at 1.0 m is 250 mm in diameter and 40 mm thick. The shaft and disk are made of steel, with a central value of mass density of ρ^c=7810 kg/m^3, a central value of the modulus of elasticity of E^c=211 GPa, and a fixed

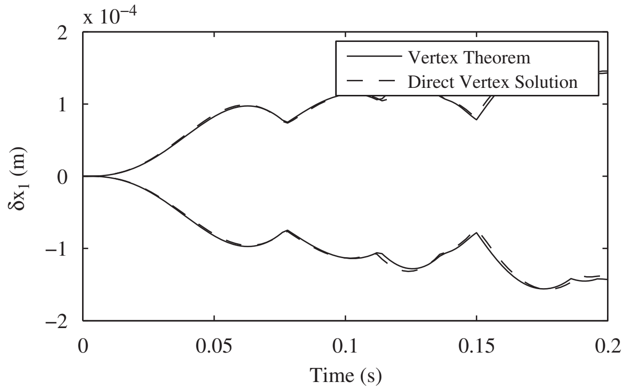

Fig. 6. Comparison of bounds of δx_1 by the vertex solution theorem for the response deviation and the direct vertex response.

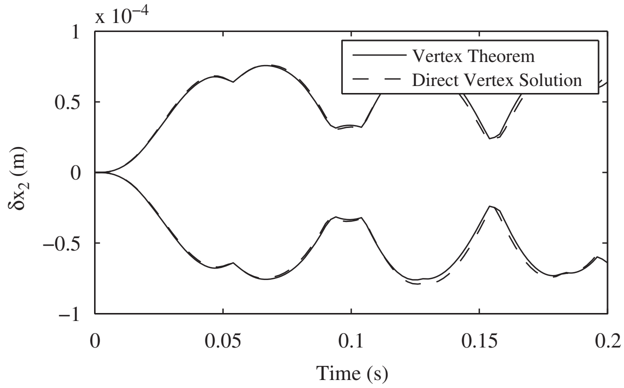

Fig. 7. Comparison of bounds of δx_2 by the vertex solution theorem for the response deviation and the direct vertex response.

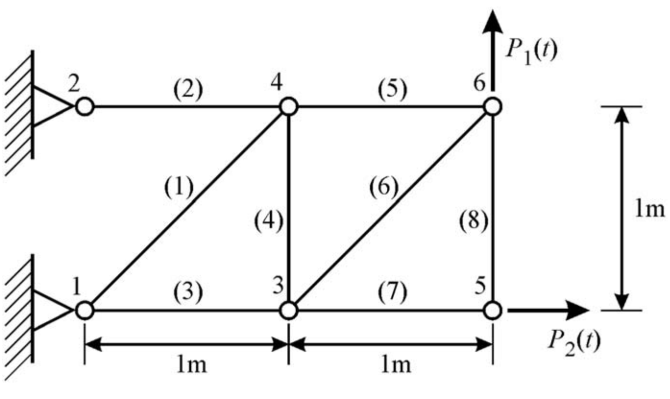

Fig. 8. A schematic of the eight-bar two-dimensional truss. The numbers indicate the nodes and the numbers in parenthesis indicate the element numbers.

Table 3
The central eigenvalues of the eight-bar truss.

$\omega_{1,2}$	$\omega_{3,4}$	$\omega_{5,6}$	$\omega_{7,8}$
$-13.73 \pm 682.52i$	$-37.84 \pm 1865.20i$	$-60.44 \pm 2510.28i$	$-103.68 \pm 3420.42i$
$\omega_{9,10}$	$\omega_{11,12}$	$\omega_{13,14}$	$\omega_{15,16}$
$-224.26 \pm 5170.27i$	$-318.073 \pm 6197.37i$	$-341.51 \pm 6428.25i$	$-402.57 \pm 6993.54i$

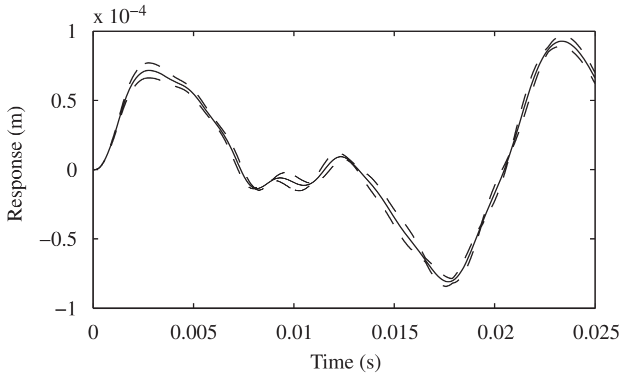

Fig. 9. The response region of the fifth node in the horizontal direction for the eight-bar truss.

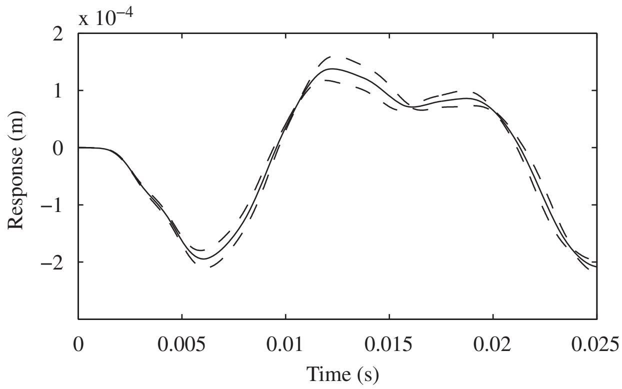

Fig. 10. The response region of the sixth node in the horizontal direction for the eight-bar truss.

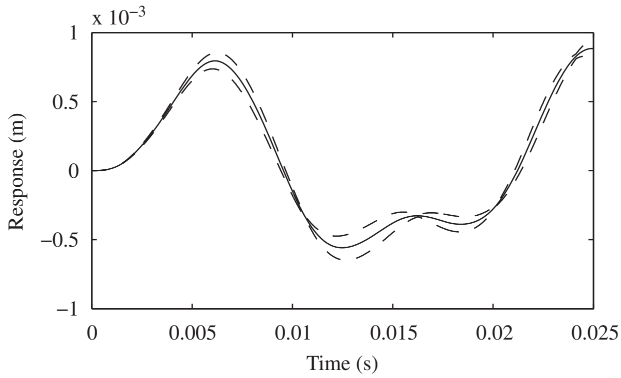

Fig. 11. The response region of the sixth node in the vertical direction for the eight-bar truss.

Poisson's ratio of 0.3. The central values of the bearing support properties are

$$k_{xx}^c = 10\,\text{MN/m},\; k_{yy}^c = 10\,\text{MN/m},\; c_{xx}^c = 400\,\text{kN s/m},\; c_{yy}^c = 400\,\text{kN s/m},$$

where x and y denote the horizontal and vertical directions, respectively. The shaft is split into three equal length elements, giving a model with four nodes and 16 degrees of freedom. An unbalance of magnitude with central value $u_b{}^c$=10^{-3} kg m acts on the disk. The central values of the initial displacements and the velocity of all the degrees of freedom are zero.

The deviations in the parameters are

$$\Delta E = 0.02E, \Delta\rho = 0.015\rho, \Delta k_{xx} = 0.02k_{xx}, \Delta k_{yy} = 0.015k_{yy}, \Delta c_{xx} = 0.02c_{xx}, \Delta c_{yy} = 0.02c_{yy}, \Delta u_b = 0.02u_b.$$

Note that the central support stiffness and damping is isotropic, but that the uncertainty in the support stiffness is anisotropic. For the initial conditions we suppose the deviations at node 2 on the shaft and at the disk are independent with maximum amplitude 0.1 mm. The machine is run-up through the first critical speed with a constant acceleration of α=0.5 Hz/s, from an initial rotor spin speed of 4 Hz. For the response calculation, the 16 degrees of freedom are reduced to four degrees of freedom using a transformation based on the lowest four modes obtained by neglecting damping and the gyroscopic effects [20]. The uncertain initial conditions need to be transformed to the reduced degrees system, although this cannot be done exactly. Thus the initial conditions are projected onto the vector space spanned by the four modes used

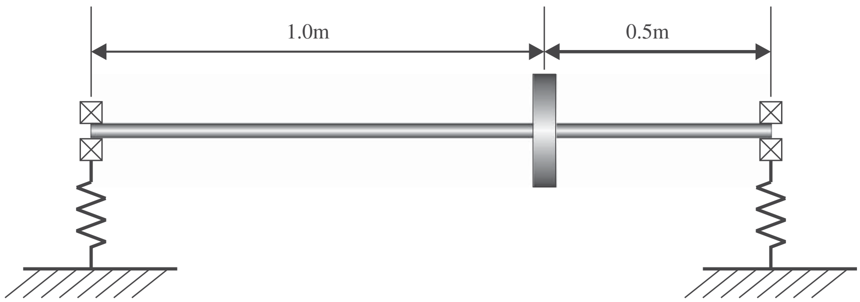

Fig. 12. The rotating machine example.

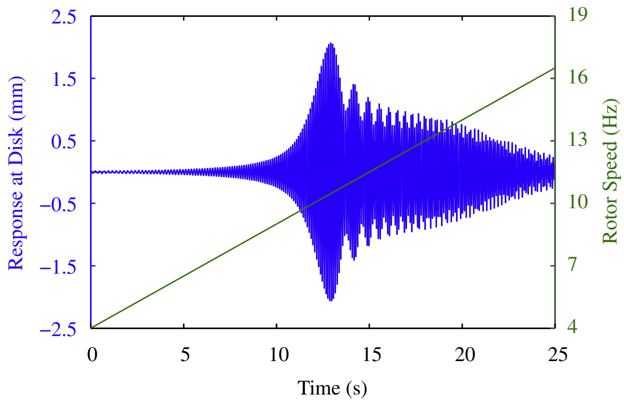

Fig. 13. The rotor speed and the central value of response at the disk.

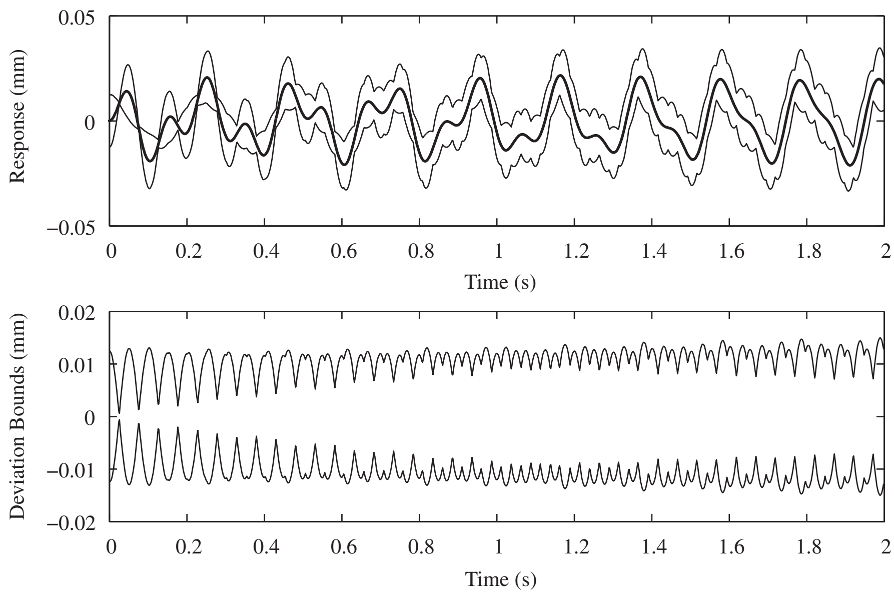

Fig. 14. The central value, infimum and supremum of the response at the disk and the deviations to central values between 0 and 2 s.

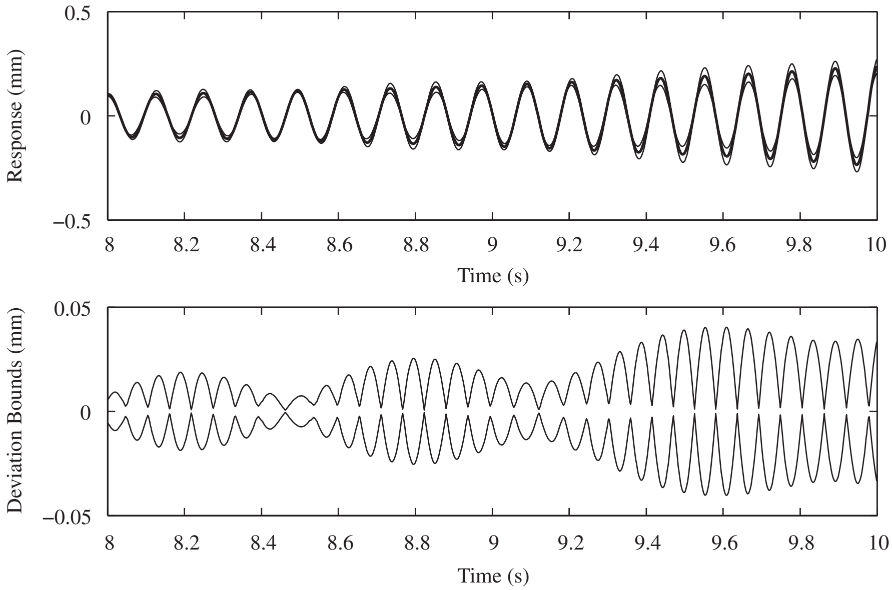

Fig. 15. The central value, infimum and supremum of the response at the disk and the deviations to central values between 8 and 10 s.

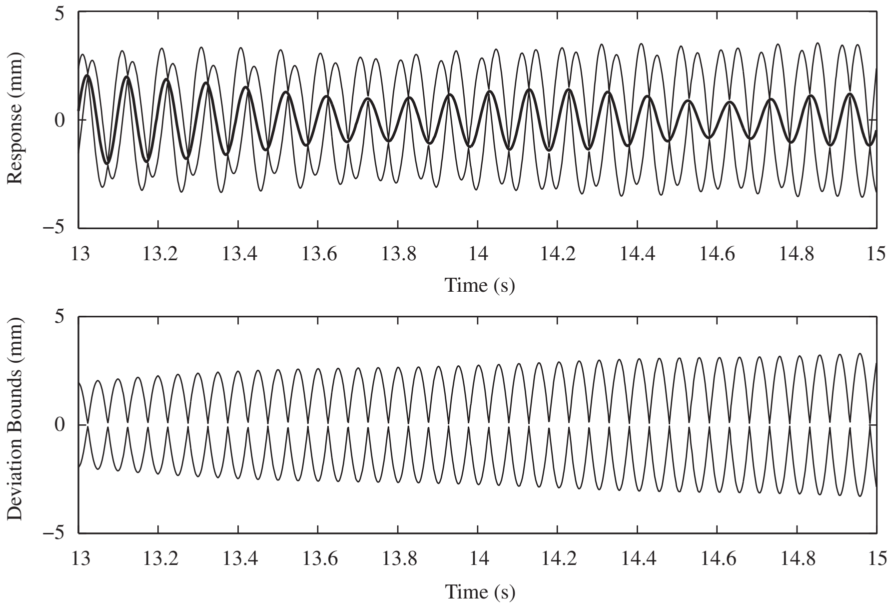

Fig. 16. The central value, infimum and supremum of the response at the disk and the deviations to central values between 13 and 15 s.

as the reduction basis. Hence the initial conditions in the reduced degrees of freedom are obtained using the Moore–Penrose pseudo-inverse of the reduction transformation.

Fig. 13 shows the rotor speed and the central value of the response at the disk. The central value, infimum and supremum of the response and the deviations from the central values at the disk over three time intervals, computed by the vertex solution theorem, are given in Figs. 14–16. These results demonstrate that the vertex solution theorem proposed in this paper is able to propagate uncertainties in the structural and external force parameters, and also uncertainties in the initial conditions.

6. Conclusions

In this paper, a new method, based on the vertex solution theorem, to determine the supremum and infimum of the first-order deviation in the dynamic response is presented. The basic idea is to convert the interval dynamic equations of motion equation into a perturbed state space form. From this state space form the exact boundaries of the first-order deviation of the dynamic response can be obtained by solving 2^{m+l} deterministic dynamic response problems, where m denotes the number of the uncertain parameters and l denotes the number of uncertain initial conditions. The method does neglect the second-order terms in the equations of motion and hence care should be exercise when the uncertain parameter intervals are large. From these responses the supremum and infimum of the dynamic responses may be easily calculated. In contrast to previous methods, the vertex solution theorem proposed in this paper allows for bounded uncertainties in the initial conditions as well as interval uncertainties in the structural parameters. Three numerical examples were used to illustrate the feasibility and the efficiency of this method, and the results were compared with the perturbation method. The numerical results show that the vertex solution theorem yields narrower bounds than those produced by the perturbation method, since interval extension is avoided. Finally, the new method does not require the calculation of the first-order derivatives of the response with respect to the structural parameters, which may be complex in many structures, although a higher number of deterministic responses must be calculated.

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